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Casimir energies and pressures for δ-function potentials

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Abstract

The Casimir energies and pressures for a massless scalar field associated with δ-function potentials in 1 + 1 and 3 + 1 dimensions are calculated. For parallel plane surfaces, the results are finite, coincide with the pressures associated with Dirichlet planes in the limit of strong coupling, and for weak coupling do not possess a power-series expansion in 1 + 1 dimension. The relation between Casimir energies and Casimir pressures is clarified, and the former are shown to involve surface terms, interpreted as the quantum vacuum energies of the surfaces. The Casimir energy for a δ-function spherical shell in 3+1 dimensions has an expression that reduces to the familiar result for a Dirichlet shell in the strong-coupling limit. However, the Casimir energy for finite coupling possesses a logarithmic divergence first appearing in third order in the weak-coupling expansion, which seems unremovable. The corresponding energies and pressures for a derivative of a δ-function potential for the same spherical geometry generalizes the TM contributions of electrodynamics. Cancellation of divergences can occur between the TE (δ-function) and TM (derivative of δ-function) Casimir energies. These results clarify recent discussions in the literature.

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1. Introduction

Since the inception of quantum mechanics, divergences associated with zero-point energy have caused a great deal of confusion. One way to deal with them was to simply define them away. This view, however, appears to be untenable, in view of the observable consequence of zero-point fluctuations in the Casimir effect, well probed experimentally [1, 2]. Calculations of such forces, and of the associated energies, are generically plagued with infinities. One modern consensus is that Casimir forces between distinct bodies may be unambiguously computed, while self-stresses (the very concept of which is only somewhat hazily understood) are typically divergent. There are some famous counterexamples: Boyer's result for the Casimir energy of a perfectly conducting spherical shell [3], and its generalizations to other geometries [4], dimensions [5, 6] and fields [7, 8]. Even situations which possess manifestly divergent energies, such as a dielectric ball [9], possess unambiguous finite dilute limits [10, 11], attributable to van der Waals forces [12].

Although these difficulties have been known since at least 1979 [9, 13–15], recently they were rediscovered and re-examined in a series of papers by the MIT group [16–21]. Perhaps more heat than light has been generated by some of the recent discussions. It is the aim of this paper to put the discussion on a somewhat clearer footing by examining Casimir energies and pressures of massless scalar fields in a δ-function potential background. (This is what the MIT group now refer to as the 'sharp' limit [19].) It is then possible to solve the problem exactly, and study how the result depends on the strength of the coupling. Although such calculations have been presented by the MIT group [17–20] based on the summation of Feynman diagrams, they seem not to have appreciated that Casimir energies for such potentials were first computed by the Leipzig group. The first calculations with planar δ-function potentials were those of Bordag *et al* [22], who found equivalent expressions for the Casimir energies given later in [19, 20]. The corresponding spherical problem was studied first by Bordag *et al* [23], who found a nonvanishing second heat kernel coefficient, indicating that the Casimir energy was divergent in third order in the coupling. After a perhaps dubious renormalization, Scandurra [24] extracted the finite part. Recently, Barton [25] has carried out related calculations, modelling a fullerene molecule to control and physically interpret the divergences, and examining the TE and TM electromagnetic modes, with conclusions not too dissimilar from those of the MIT group.

Although this model seems quite well studied, it is perhaps worthwhile to re-examine it in what I consider the most physically transparent Green function approach, to see if some clarity can be brought to what seems at present a rather confused situation[1]. In doing so, we shall clarify the discussion of the perturbative expansion, and learn that it is only the strong-coupling limit of the spherical Casimir energy that possesses a finite self-stress, unless cancellations can occur between TE and TM modes (which certainly do occur in the strong coupling limit).

This paper is laid out as follows. In the next section, we find the Casimir pressure for a massless scalar interacting with two δ-function potentials in one spatial dimension. (Equivalently, this is a spherical geometry in one dimension.) The pressure is completely finite, but is nonanalytic in the coupling for weak coupling. The Casimir energy receives contributions from the boundaries (surface terms). The generalization to δ-function planes in three dimensions is immediate, and given in section 3. Section 4 presents the corresponding calculation for the Casimir energy of a massless scalar interacting with a spherical δ-function shell. That resulting expression, in the strong-coupling limit, reduces to the standard one for a Dirichlet shell, yielding a finite self-energy [26]. However, for any finite coupling, the expression possesses an irremovable logarithmic divergence, which first appears in third order in the weak-coupling expansion [19, 20, 23], although in second order, as noted previously [26], the energy is finite. Section 5 presents the Casimir energy and pressure for a spherical derivative of a δ-function potential, which, in the strong coupling limit, corresponds to the TM modes of electrodynamics. (The Dirichlet modes computed in section 4 correspond to the

[1] Barton [25] refers to my approach as 'older methods,' but he employs methods of Debye going back to early in the previous century, and other classic techniques. I certainly feel in good company if I use the propagation functions invented by Green, as well as Debye expansions.

TE modes.) Concluding remarks are made in section 6. The meaning of the surface terms is discussed in the appendices.

2. 1 + 1 Dimensions

We consider a massive scalar field (mass μ) interacting with two δ-function potentials, one at $x = 0$ and the other at $x = a$, which has an interaction Lagrange density

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\lambda\delta(x)\phi^2(x) - \tfrac{1}{2}\lambda'\delta(x-a)\phi^2(x) \tag{2.1}$$

where we note that the coupling constants λ and λ' have dimensions of mass. The Casimir energy for this situation may be computed in terms of the Green function G,

$$G(x, x') = \mathrm{i}\langle T\phi(x)\phi(x')\rangle \tag{2.2}$$

which has a time Fourier transform,

$$G(x, x') = \int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')} g(x, x'; \omega) \tag{2.3}$$

which in turn satisfies

$$\left[-\frac{\partial^2}{\partial x^2} + \kappa^2 + \lambda\delta(x) + \lambda'\delta(x-a)\right] g(x, x') = \delta(x - x'). \tag{2.4}$$

Here $\kappa^2 = \mu^2 - \omega^2$. This equation is easily solved, with the result[2]

$$\begin{aligned} g(x, x') = \frac{1}{2\kappa}\,\mathrm{e}^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta}\bigg[&\frac{\lambda\lambda'}{(2\kappa)^2} 2\cosh\kappa|x - x'| \\ &- \frac{\lambda}{2\kappa}\left(1 + \frac{\lambda'}{2\kappa}\right)\mathrm{e}^{2\kappa a}\,\mathrm{e}^{-\kappa(x+x')} - \frac{\lambda'}{2\kappa}\left(1 + \frac{\lambda}{2\kappa}\right)\mathrm{e}^{\kappa(x+x')}\bigg] \end{aligned} \tag{2.5a}$$

for both fields inside, $0 < x, x' < a$, while if both field points are outside, $a < x, x'$,

$$g(x, x') = \frac{1}{2\kappa}\,\mathrm{e}^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta}\,\mathrm{e}^{-\kappa(x+x'-2a)}\left[-\frac{\lambda}{2\kappa}\left(1 - \frac{\lambda'}{2\kappa}\right) - \frac{\lambda'}{2\kappa}\left(1 + \frac{\lambda}{2\kappa}\right)\mathrm{e}^{2\kappa a}\right]. \tag{2.5b}$$

For $x, x' < 0$,

$$g(x, x') = \frac{1}{2\kappa}\,\mathrm{e}^{-\kappa|x-x'|} + \frac{1}{2\kappa\Delta}\,\mathrm{e}^{\kappa(x+x')}\left[-\frac{\lambda'}{2\kappa}\left(1 - \frac{\lambda}{2\kappa}\right) - \frac{\lambda}{2\kappa}\left(1 + \frac{\lambda'}{2\kappa}\right)\mathrm{e}^{2\kappa a}\right]. \tag{2.5c}$$

Here, the denominator is

$$\Delta = \left(1 + \frac{\lambda}{2\kappa}\right)\left(1 + \frac{\lambda'}{2\kappa}\right)\mathrm{e}^{2\kappa a} - \frac{\lambda\lambda'}{(2\kappa)^2}. \tag{2.6}$$

Note that in the strong coupling limit we recover the familiar results, for example, inside

$$\lambda, \lambda' \to \infty: \quad g(x, x') \to -\frac{\sinh\kappa x_< \sinh\kappa(x_> - a)}{\kappa\sinh\kappa a}. \tag{2.7}$$

We can now calculate the force at the one-loop level on one of the δ-function points by calculating the discontinuity of the stress tensor, obtained from the Green function by

$$\langle T^{\mu\nu}\rangle = \left(\partial^\mu\partial^{\nu\prime} - \frac{1}{2}g^{\mu\nu}\partial^\lambda\partial'_\lambda\right)\frac{1}{\mathrm{i}}G(x, x')\bigg|_{x=x'}. \tag{2.8}$$

[2] The Green function may also be readily derived from the multiple reflection formalism given in [27], in terms of the reflection amplitude for a single interface, $r = -(1 + 2\kappa/\lambda)^{-1}$.

Writing

$$\langle T^{\mu\nu}\rangle = \int \frac{d\omega}{2\pi} t^{\mu\nu} \tag{2.9}$$

we find inside

$$\begin{aligned} t_{xx} &= \frac{1}{2i}(\omega^2 + \partial_x \partial_{x'}) g(x, x')\Big|_{x=x'} \\ &= \frac{1}{4i\kappa\Delta}\left\{(2\omega^2 - \mu^2)\left[\left(1 + \frac{\lambda}{2\kappa}\right)\left(1 + \frac{\lambda'}{2\kappa}\right) e^{2\kappa a} + \frac{\lambda\lambda'}{(2\kappa)^2}\right]\right. \\ &\quad \left. - \mu^2 \left[\frac{\lambda}{2\kappa}\left(1 + \frac{\lambda'}{2\kappa}\right) e^{-2\kappa(x-a)} + \frac{\lambda'}{2\kappa}\left(1 + \frac{\lambda}{2\kappa}\right) e^{2\kappa x}\right]\right\}. \end{aligned} \tag{2.10}$$

Let us henceforth simplify the considerations by taking the massless limit, $\mu = 0$. Note then that the conformal invariance of the free theory is reflected in the tracelessness of $t^{\mu\nu}$,

$$\langle T^{\mu}{}_{\mu}\rangle = 0 \quad \Rightarrow \quad t^{00} = t_{xx}. \tag{2.11}$$

The stress tensor just to the left of the point $x = a$ is

$$t_{xx}\Big|_{x=a-} = -\frac{\kappa}{2i}\left\{1 + 2\left[\left(\frac{2\kappa}{\lambda} + 1\right)\left(\frac{2\kappa}{\lambda'} + 1\right) e^{2\kappa a} - 1\right]^{-1}\right\}. \tag{2.12}$$

From this we must subtract the stress just to the right of the point at $x = a$, obtained from equation (2.5b), which turns out to be in the massless limit

$$t_{xx}\Big|_{x=a+} = -\frac{\kappa}{2i} \tag{2.13}$$

which just cancels the 1 in braces in equation (2.12). Thus the force at the point $x = a$ due to the quantum fluctuations in the scalar field is given by the simple, finite expression

$$F = \langle T_{xx}\rangle\Big|_{x=a-} - \langle T_{xx}\rangle\Big|_{x=a+} = -\frac{1}{4\pi a^2}\int_0^\infty dy\, y \frac{1}{(y/\lambda a + 1)(y/\lambda' a + 1)\, e^y - 1}. \tag{2.14}$$

This reduces to the well-known Lüscher result [28, 29] in the limit $\lambda, \lambda' \to \infty$,

$$\lim_{\lambda=\lambda'\to\infty} F = -\frac{\pi}{24a^2} \tag{2.15}$$

and for $\lambda = \lambda'$ is plotted in figure 1.

We can also compute the energy density. In this simple massless case, the calculation appears identical, because $t_{xx} = t_{00}$. The energy density is constant (equation (2.10) with $\mu = 0$) and subtracting from it the a-independent part that would be present if no potential were present, we immediately see that the total energy is $E = Fa$, so $F = -\partial E/\partial a$ (holding the dimensionless coupling λa constant). This result differs significantly from that given in [17, 18, 21], which is a divergent expression in the massless limit, not transformable into the expression found by this naive procedure. However, that result may be easily derived from the following expression for the total energy[3]:

$$\begin{aligned} E &= \int (d\mathbf{r})\langle T^{00}\rangle = \frac{1}{2i}\int (d\mathbf{r})(\partial^0\partial'^0 - \nabla^2) G(x, x')\Big|_{x=x'} \\ &= \frac{1}{2i}\int (d\mathbf{r}) \int \frac{d\omega}{2\pi} 2\omega^2 \mathcal{G}(\mathbf{r}, \mathbf{r}). \end{aligned} \tag{2.16}$$

[3] This is a formal expression which needs to be regulated, for example, by point splitting or by dimensional continuation, both of which are explicitly considered in [26]. See appendix A for further discussion.

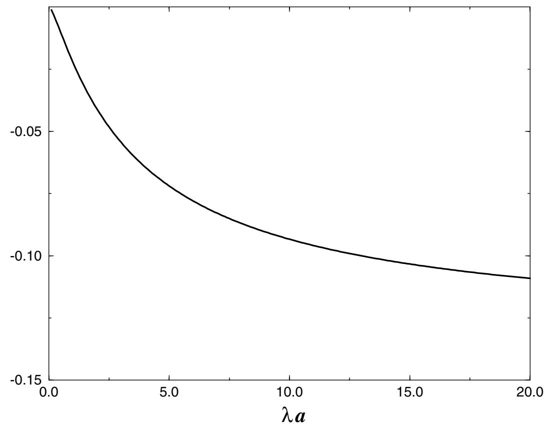

Figure 1. Casimir force (2.14) between two δ-function points having strength λ and separated by a distance a.

Integrating over the Green functions in the three regions, given by equations (2.5a), (2.5b) and (2.5c), we obtain for $\lambda = \lambda'$,

$$E = \frac{1}{2\pi a}\int_0^\infty \mathrm{d}y \frac{1}{1+y/\lambda a} - \frac{1}{4\pi a}\int_0^\infty \mathrm{d}y\, y \frac{1+2/(y+\lambda a)}{(y/\lambda a+1)^2 e^y - 1} \tag{2.17}$$

where the first term, which is twice the surface energy of a single interface, as shown in appendix B, is regarded as an irrelevant constant (λ is constant), and the second is the same as that given by equation (70) of [17] upon integration by parts.

The origin of this discrepancy is the existence of a surface contribution to the energy. Because $\partial_\mu T^{\mu\nu} = 0$, we have, for a region V bounded by a surface S,

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_V (\mathrm{d}\mathbf{r}) T^{00} + \oint_S \mathrm{d}S_i T^{0i}. \tag{2.18}$$

Here $T^{0i} = \partial^0\phi\partial^i\phi$, so we conclude that there is an additional contribution to the energy,

$$E_s = -\frac{1}{2\mathrm{i}}\int \mathrm{d}\mathbf{S}\cdot\nabla G(x,x')\bigg|_{x'=x} \tag{2.19a}$$

$$= -\frac{1}{2\mathrm{i}}\int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi}\sum \frac{\mathrm{d}}{\mathrm{d}x} g(x,x')\bigg|_{x'=x} \tag{2.19b}$$

where the derivative is taken at the boundaries (here $x = 0, a$) in the sense of the outward normal from the region in question. In this case

$$\frac{\mathrm{d}}{\mathrm{d}x}g\bigg|_{x=x'=a-} - \frac{\mathrm{d}}{\mathrm{d}x}g\bigg|_{x=x'=a+} = \frac{\mathrm{d}}{\mathrm{d}x}g\bigg|_{x=x'=0-} - \frac{\mathrm{d}}{\mathrm{d}x}g\bigg|_{x=x'=0+}$$
$$= \frac{\lambda}{2\kappa\Delta}\left[\frac{\lambda}{2\kappa} - \left(1+\frac{\lambda}{2\kappa}\right)\mathrm{e}^{2\kappa a}\right] \tag{2.20}$$

so when this is inserted into (2.19b), we obtain

$$E_s = \frac{1}{2\pi a}\int_0^\infty \mathrm{d}y \frac{1}{y/\lambda a + 1} - \frac{1}{2\pi a}\int_0^\infty \frac{\mathrm{d}y\, y}{y+\lambda a}\frac{1}{(y/\lambda a+1)^2\,\mathrm{e}^y - 1} \tag{2.21}$$

precisely the surface terms in (2.17). The integrated formula (2.16) automatically builds in this surface contribution, as the implicit surface term in the integration by parts. (These terms are slightly unfamiliar because they do not arise in the cases of Neumann or Dirichlet boundary conditions.) See Fulling [30] for further discussion; see also the appendices.

It is interesting to consider the behaviour of the force or energy for small coupling λ. It is clear that, in fact, equation (2.14) is not analytic at $\lambda = 0$. (This reflects an infrared divergence in the Feynman diagram calculation.) If we extract the leading λ^2 term we are left with a divergent integral. A correct asymptotic evaluation leads to the behaviour

$$F \sim \frac{\lambda^2}{4\pi}(\ln 2\lambda a + \gamma) \qquad E \sim -\frac{\lambda^2 a}{4\pi}(\ln 2\lambda a + \gamma - 1) \qquad \lambda \to 0. \tag{2.22}$$

This behaviour indeed was anticipated in earlier perturbative analyses. In [26], the general result was given for the Casimir energy for a D-dimensional spherical δ-function potential (a factor of $1/4\pi$ was inadvertently omitted, and $g = \lambda a$)

$$E = -2^{-1-2D}\frac{\lambda^2 a}{\pi}\frac{\Gamma\left(\frac{D-1}{2}\right)\Gamma(D-3/2)\Gamma(1-D/2)}{[\Gamma(D/2)]^2}. \tag{2.23}$$

This possesses an infrared divergence as $D \to 1$:

$$E^{(D=1)} = \frac{\lambda^2 a}{4\pi}\Gamma(0) \tag{2.24}$$

which is consistent with the nonanalytic behaviour seen in equation (2.22).

3. Parallel planes in 3 + 1 dimensions

It is trivial to extract the expression for the Casimir pressure between two δ function planes in three spatial dimensions, where the background lies at $x = 0$ and $x = a$. We merely have to insert into the above a transverse momentum transform,

$$G(x, x') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \int \frac{(d\mathbf{k})}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')_\perp} g(x, x'; \kappa) \tag{3.1}$$

where now $\kappa^2 = \mu^2 + k^2 - \omega^2$. Then g has exactly the same form as in equations (2.5). The reduced stress tensor is given by, for the massless case[4],

$$t_{xx} = \frac{1}{2}(\partial_x \partial_{x'} - \kappa^2)\frac{1}{i}g(x, x')\bigg|_{x=x'}. \tag{3.2}$$

So we immediately see that the attractive pressure on the planes is given by ($\lambda = \lambda'$)

$$P = -\frac{1}{32\pi^2 a^4}\int_0^\infty dy\, y^3 \frac{1}{(y/\lambda a + 1)^2 e^y - 1} \tag{3.3}$$

which coincides with the result given in [19, 20].

The Casimir energy per unit area again might be expected to be

$$\mathcal{E} = -\frac{1}{96\pi^2 a^3}\int_0^\infty dy \frac{y^3}{(y/\lambda a + 1)^2 e^y - 1} = \frac{1}{3}\frac{P}{a} \tag{3.4}$$

because then, naively $P = -\frac{\partial}{\partial a}\mathcal{E}$, if λa is held fixed. In fact, it is straightforward to compute the energy density $\langle T^{00}\rangle$ is the three regions, $x < 0$, $0 < x < a$ and $a < x$, and then integrate it over x to obtain the energy/area, which differs from equation (3.4) because, now, there exists

[4] Use of the conformal rather than the canonical stress tensor is without effect for calculating the pressure or the total observable energy. See [2, 26].

transverse momentum. We must also include the surface term (2.19*a*), which is of opposite sign, and of double magnitude, compared to the k^2 term. The net extra term is

$$\mathcal{E}' = \frac{1}{48\pi^2 a^3}\int_0^\infty \mathrm{d}y\, y^2 \frac{1}{1+y/\lambda a}\left[1 - \frac{y/\lambda a}{(y/\lambda a+1)^2\,\mathrm{e}^y - 1}\right]. \tag{3.5}$$

If we regard λ as constant (so that the strength of the coupling is independent of the separation between the planes) we may drop the first, divergent term here as irrelevant, being independent of a, because $y = 2\kappa a$, and then the total energy is

$$\mathcal{E} = -\frac{1}{96\pi^2 a^3}\int_0^\infty \mathrm{d}y\, y^3 \frac{1 + 2/(\lambda a + y)}{(y/\lambda a + 1)^2\,\mathrm{e}^y - 1} \tag{3.6}$$

which coincides with the massless limit of the energy first found by Bordag *et al* [22], and given in [19, 20]. As noted in section 2, this result may also readily be derived using (2.16). When differentiated with respect to a, equation (3.6), with λ fixed, yields the pressure (3.3).

The modes considered here correspond, in the $\lambda \to \infty$ limit, to TE modes for perfectly conducting planes. For the modes corresponding to TM modes, see [27].

4. Three-dimensional spherical potential

We now carry out the same calculation in three spatial dimensions, with a radially symmetric background

$$\mathcal{L}_{\text{int}} = -\tfrac{1}{2}\lambda\delta(r-a)\phi^2(x) \tag{4.1}$$

which would correspond to a Dirichlet shell in the limit $\lambda \to \infty$. The time-Fourier transformed Green function satisfies the equation ($\kappa^2 = -\omega^2$)

$$[-\nabla^2 + \kappa^2 + \lambda\delta(r-a)]\mathcal{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'). \tag{4.2}$$

We write $\mathcal{G}$ in terms of a reduced Green function

$$\mathcal{G}(\mathbf{r},\mathbf{r}') = \sum_{lm} g_l(r,r') Y_{lm}(\Omega) Y^*_{lm}(\Omega') \tag{4.3}$$

where g_l satisfies

$$\left[-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + \kappa^2 + \lambda\delta(r-a)\right] g_l(r,r') = \frac{1}{r^2}\delta(r-r'). \tag{4.4}$$

We solve this in terms of modified Bessel functions, $I_\nu(x)$, $K_\nu(x)$, where $\nu = l + 1/2$, which satisfy the Wronskian condition

$$I'_\nu(x)K_\nu(x) - K'_\nu(x)I_\nu(x) = \frac{1}{x}. \tag{4.5}$$

We solve equation (4.4) by requiring continuity of g_l at each singularity, r' and a, and the appropriate discontinuity of the derivative. Inside the sphere we then find ($0 < r, r' < a$)

$$g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{\lambda}{\kappa}s_l(\kappa r)s_l(\kappa r')\frac{e_l^2(\kappa a)}{1 + \frac{\lambda}{\kappa}s_l(\kappa a)e_l(\kappa a)}\right]. \tag{4.6}$$

Here we have introduced the modified Riccati–Bessel functions,

$$s_l(x) = \sqrt{\frac{\pi x}{2}}I_{l+1/2}(x) \qquad e_l(x) = \sqrt{\frac{2x}{\pi}}K_{l+1/2}(x). \tag{4.7}$$

Note that equation (4.6) reduces to the expected result, vanishing as $r \to a$, in the limit of strong coupling:

$$\lim_{\lambda\to\infty} g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{e_l(\kappa a)}{s_l(\kappa a)}s_l(\kappa r)s_l(\kappa r')\right]. \tag{4.8}$$

When both points are outside the sphere, $r, r' > a$, we obtain a similar result:

$$g_l(r,r') = \frac{1}{\kappa r r'}\left[e_l(\kappa r_>)s_l(\kappa r_<) - \frac{\lambda}{\kappa}e_l(\kappa r)e_l(\kappa r')\frac{s_l^2(\kappa a)}{1+\frac{\lambda}{\kappa}s_l(\kappa a)e_l(\kappa a)}\right] \tag{4.9}$$

which similarly reduces to the expected result as $\lambda \to \infty$.

Now we want to get the radial–radial component of the stress tensor to get the pressure on the sphere, which is obtained by applying the operator

$$\partial_r\partial_{r'} - \frac{1}{2}(-\partial^0\partial'^0 + \nabla\cdot\nabla') \to \frac{1}{2}\partial_r\partial_{r'} - \kappa^2 - \frac{l(l+1)}{r^2} \tag{4.10}$$

to the Green function, where in the last term we have averaged over the surface of the sphere. In this way we find, from the discontinuity of $\langle T_{rr}\rangle$ across the $r = a$ surface, the net stress

$$\mathcal{S} = \frac{\lambda}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\,\frac{(e_l(x)s_l(x))' - \frac{2e_l(x)s_l(x)}{x}}{1+\frac{\lambda a e_l(x)s_l(x)}{x}}. \tag{4.11}$$

The same result can be deduced by computing the total energy (2.16). The free Green function, the first term in equation (4.6) or (4.9), evidently makes no significant contribution to the energy, for it gives a term independent of the radius of the sphere, a, so we omit it. The remaining radial integrals are simply

$$\int_0^x \mathrm{d}y\, s_l^2(y) = \frac{1}{2x}\left[(x^2 + l(l+1))s_l^2 + x s_l s_l' - x^2 s_l'^2\right] \tag{4.12a}$$

$$\int_x^\infty \mathrm{d}y\, e_l^2(y) = -\frac{1}{2x}\left[(x^2 + l(l+1))e_l^2 + x e_l e_l' - x^2 e_l'^2\right] \tag{4.12b}$$

where all the Bessel functions on the right-hand sides of these equations are evaluated at x. Then using the Wronskian, we find that the Casimir energy is

$$E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\ln[1+\lambda a I_\nu(x)K_\nu(x)]. \tag{4.13}$$

If we differentiate with respect to a, with λ fixed, we immediately recover the stress (4.11). This expression, upon integration by parts, coincides with that given by Barton [25], and was first analysed in detail by Scandurra [24]. It reduces to the well-known expression for the Casimir energy of a massless scalar field inside and outside a sphere upon which Dirichlet boundary conditions are imposed, that is, the field must vanish at $r = a$:

$$\lim_{\lambda\to\infty} E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\ln[I_\nu(x)K_\nu(x)] \tag{4.14}$$

because multiplying the argument of the logarithm by a power of x is without effect, corresponding to a contact term. Details of the evaluation of equation (4.14) are given in [26].

The opposite limit is of interest here. The expansion of the logarithm is immediate for small λ. The first term, of order λ, is evidently divergent, but irrelevant, since that may be removed by renormalization of the tadpole graph. In contradistinction to the claim of [17–20], the order λ^2 term is finite, as claimed in [26]. That term is[5]

$$E^{(\lambda^2)} = \frac{\lambda^2 a}{4\pi}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\,\mathrm{x}\frac{\mathrm{d}}{\mathrm{d}x}[I_{l+1/2}(x)K_{l+1/2}(x)]^2. \tag{4.15}$$

[5] It is very interesting that if (4.15) is integrated by parts, and the boundary term at $x = \infty$ is omitted, the result of [17–20] is recovered. However, the latter expression is divergent, as we see below, while the former, which is the result directly obtained by the approach given here, is finite. Field theory is more than a set of Feynman rules.

The sum on l can be carried out using a trick due to Klich [31]: the sum rule

$$\sum_{l=0}^{\infty}(2l+1)e_l(x)s_l(y)P_l(\cos\theta) = \frac{xy}{\rho}\,\mathrm{e}^{-\rho} \tag{4.16}$$

where $\rho = \sqrt{x^2+y^2-2xy\cos\theta}$, is squared, and then integrated over θ, according to

$$\int_{-1}^{1}\mathrm{d}\cos\theta\,P_l(\cos\theta)P_{l'}(\cos\theta) = \delta_{ll'}\frac{2}{2l+1}. \tag{4.17}$$

In this way we learn that

$$\sum_{l=0}^{\infty}(2l+1)e_l^2(x)s_l^2(x) = \frac{x^2}{2}\int_0^{4x}\frac{\mathrm{d}w}{w}\,\mathrm{e}^{-w}. \tag{4.18}$$

Although this integral is divergent, because we did not integrate by parts in equation (4.15), that divergence does not contribute:

$$E^{(\lambda^2)} = \frac{\lambda^2 a}{4\pi}\int_0^{\infty}\mathrm{d}x\,\frac{1}{2}x\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{4x}\frac{dw}{w}\,\mathrm{e}^{-w} = \frac{\lambda^2 a}{32\pi} \tag{4.19}$$

which is exactly the result (4.25) of [26], which also follows from equation (2.23) here.

However, before we wax too euphoric, we recognize that the order λ^3 term appears logarithmically divergent, just as [19] and [20] claim. This does not signal a breakdown in perturbation theory, as the divergence in the $D = 1$ calculation did. Suppose we subtract off the two leading terms,

$$E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^{\infty}\mathrm{d}x\,x\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(1+\lambda a I_\nu K_\nu\right) - \lambda a I_\nu K_\nu + \frac{\lambda^2 a^2}{2}(I_\nu K_\nu)^2\right] + \frac{\lambda^2 a}{32\pi}. \tag{4.20}$$

To study the behaviour of the sum for large values of l, we can use the uniform asymptotic expansion (Debye expansion),

$$\nu \gg 1: \quad I_\nu(x)K_\nu(x) \sim \frac{t}{2\nu}\left[1 + \frac{A(t)}{\nu^2} + \frac{B(t)}{\nu^4} + \cdots\right]. \tag{4.21}$$

Here $x = \nu z$, and $t = 1/\sqrt{1+z^2}$. The functions A, B, etc, are polynomials in t. We now insert this into equation (4.20) and expand not in λ but in ν; the leading term is

$$E^{(\lambda^3)} \sim \frac{\lambda^3 a^2}{24\pi}\sum_{l=0}^{\infty}\frac{1}{\nu}\int_0^{\infty}\frac{\mathrm{d}z}{(1+z^2)^{3/2}} = \frac{\lambda^3 a^2}{24\pi}\zeta(1). \tag{4.22}$$

Although the frequency integral is finite, the angular momentum sum is divergent. The appearance here of the divergent $\zeta(1)$ seems to signal an insuperable barrier to extraction of a finite Casimir energy for finite λ.

This divergence has been known for many years, and was first calculated explicitly in 1998 by Bordag *et al* [23], where the second heat kernel coefficient gave

$$E \sim \frac{\lambda^3 a^2}{48\pi}\frac{1}{s} \qquad s \to 0 \tag{4.23}$$

which exactly corresponds to (4.22). A possible way of dealing with this divergence was advocated in [24].

5. TM spherical potential

Of course, the scalar model considered in the previous section is merely a toy model, and something analogous to electrodynamics is of far more physical relevance. There are good reasons for believing that cancellations occur in general between TE (Dirichlet) and TM (Robin) modes. Certainly they do occur in the classic Boyer energy of a perfectly conducting spherical shell [3, 32, 33], and the indications are that such cancellations occur even with imperfect boundary conditions [25]. Following the latter reference, let us consider the potential (λ now has dimensions of length)

$$\mathcal{L}_{\text{int}} = \frac{1}{2}\lambda\frac{1}{r}\frac{\partial}{\partial r}\delta(r-a)\phi^2(x). \tag{5.1}$$

In the limit $\lambda \to \infty$ this corresponds to TM boundary conditions. The reduced Green function is thus taken to satisfy

$$\left[-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}r^2\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{r^2} + \kappa^2 - \frac{\lambda}{r}\frac{\partial}{\partial r}\delta(r-a)\right]g_l(r,r') = \frac{1}{r^2}\delta(r-r'). \tag{5.2}$$

At $r = r'$ we have the usual boundary conditions, that g_l be continuous, but that its derivative be discontinuous,

$$r^2\frac{\mathrm{d}}{\mathrm{d}r}g_l\bigg|_{r=r'-}^{r=r'+} = -1 \tag{5.3}$$

while at the surface of the sphere the derivative is continuous,

$$\frac{\partial}{\partial r}rg_l\bigg|_{r=a-}^{r=a+} = 0 \tag{5.4a}$$

while the function is discontinuous,

$$g_l\bigg|_{r=a-}^{r=a+} = -\frac{\lambda}{a}\frac{\partial}{\partial r}rg_l. \tag{5.4b}$$

It is then easy to find the Green function. When both points are inside the sphere,

$$r, r' < a: \quad g_l(r,r') = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<)e_l(\kappa r_>) - \frac{\lambda\kappa[e_l'(\kappa a)]^2 s_l(\kappa r)s_l(\kappa r')}{1+\lambda\kappa e_l'(\kappa a)s_l'(\kappa a)}\right] \tag{5.5a}$$

and when both points are outside the sphere,

$$r, r' > a: \quad g_l(r,r') = \frac{1}{\kappa r r'}\left[s_l(\kappa r_<)e_l(\kappa r_>) - \frac{\lambda\kappa[s_l'(\kappa a)]^2 e_l(\kappa r)e_l(\kappa r')}{1+\lambda\kappa e_l'(\kappa a)s_l'(\kappa a)}\right]. \tag{5.5b}$$

These supply the appropriate Robin boundary conditions in the $\lambda \to \infty$ limit:

$$\lim_{\lambda\to 0}\frac{\partial}{\partial r}rg_l\bigg|_{r=a} = 0. \tag{5.6}$$

The Casimir energy may be readily obtained from equation (2.16), and we find, using the integrals (4.12),

$$E = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[1+\frac{\lambda}{a}xe_l'(x)s_l'(x)\right]. \tag{5.7}$$

The stress may be obtained from this by applying $-\partial/\partial a$, and regarding λ as constant, or directly, from the Green function by applying a differential operator,

$$t_{rr} = \frac{1}{2\mathrm{i}}\left[\nabla_r\nabla_{r'} - \kappa^2 - \frac{l(l+1)}{r^2}\right] g_l\bigg|_{r'=r} \tag{5.8}$$

which is the same as that in equation (4.10), except that

$$\nabla_r = \frac{1}{r}\partial_r r \tag{5.9}$$

appropriate to TM boundary conditions (see [6], for example). Either way, the total stress on the sphere is

$$\mathcal{S} = -\frac{\lambda}{2\pi a^3}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x^2 \frac{[e_l'(x)s_l'(x)]'}{1+(\lambda/a)xe_l'(x)s_l'(x)}. \tag{5.10}$$

The result for the energy (5.7) is similar, but not identical, to that given by Barton [25].

Suppose we now combine the TE and TM Casimir energies, equations (4.13) and (5.7):

$$E^{\mathrm{TE}} + E^{\mathrm{TM}} = -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\ln\left[\left(1+\lambda^{\mathrm{TE}}a\frac{e_l s_l}{x}\right)\left(1+\frac{\lambda^{\mathrm{TM}}}{a}xe_l's_l'\right)\right]. \tag{5.11}$$

In the limit $\lambda^{\mathrm{TE,TM}} \to \infty$ this reduces to the familiar expression for the perfectly conducting spherical shell [32]:

$$\lim_{\lambda\to\infty} E = -\frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\left(\frac{e_l'}{e_l}+\frac{e_l''}{e_l'}+\frac{s_l'}{s_l}+\frac{s_l''}{s_l'}\right). \tag{5.12}$$

Here we have, as appropriate to the electrodynamic situation, omitted the $l = 0$ mode. This expression yields a finite Casimir energy. What about finite λ? In general, it appears that there is no chance that the divergence found in the previous section in order λ^3 can be cancelled. But suppose the couplings for the TE and TM modes are different. If $\lambda^{\mathrm{TE}}\lambda^{\mathrm{TM}} = 4$, a cancellation appears possible.

Let us illustrate this by retaining only the leading terms in the uniform asymptotic expansions: $(x = \nu z)$

$$\frac{e_l(x)s_l(x)}{x} \sim \frac{t}{2\nu} \qquad xe_l'(x)s_l'(x) \sim -\frac{\nu}{2t} \qquad \nu\to\infty. \tag{5.13}$$

Then the logarithm appearing in the integral for the energy (5.11) is approximately

$$\ln \sim \ln\left(-\frac{\lambda^{\mathrm{TM}}\nu}{2at}\right) + \ln\left(1+\frac{\lambda^{\mathrm{TE}}at}{2\nu}\right) + \ln\left(1-\frac{2at}{\lambda^{\mathrm{TM}}\nu}\right). \tag{5.14}$$

The first term here presumably gives no contribution to the energy, because it is independent of λ upon differentiation, and further we may interpret $\sum_{l=0}^{\infty}\nu^2 = 0$ (see equation (5.18)). Now if we make the above identification of the couplings,

$$\hat{\lambda} = \frac{\lambda^{\mathrm{TE}}a}{2} = \frac{2a}{\lambda^{\mathrm{TM}}} \tag{5.15}$$

all the odd powers of ν cancel out, and

$$E \sim -\frac{1}{2\pi a}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(1-\frac{\hat{\lambda}^2t^2}{\nu^2}\right). \tag{5.16}$$

The divergence encountered for the TE mode is thus removed, and the power series is simply twice the sum of the even terms there. This will be finite. Presumably, the same is true if the subleading terms in the uniform asymptotic expansion are retained.

It is interesting to approximately evaluate equation (5.16). The integral over z may be easily evaluated as a contour integral, leaving

$$E \sim -\frac{1}{a}\sum_{l=0}^{\infty} \nu^2 \left(1 - \sqrt{1 - \frac{\hat{\lambda}^2}{\nu^2}}\right). \tag{5.17}$$

This l sum is logarithmically divergent, an artefact of the asymptotic expansion, since we know the λ^2 term is finite. If we expand the square root for small $\hat{\lambda}^2/\nu^2$, we see that the $\mathcal{O}(\hat{\lambda}^2)$ term vanishes if we interpret the sum as

$$\sum_{l=0}^{\infty} \nu^{-s} = (2^s - 1)\zeta(s) \tag{5.18}$$

in terms of the Riemann zeta function. The leading term is $\mathcal{O}(\hat{\lambda}^4)$:

$$E \sim -\frac{\hat{\lambda}^4}{8a}\sum_{l=0}^{\infty}\frac{1}{\nu^2} = \frac{\hat{\lambda}^4\pi^2}{16a}. \tag{5.19}$$

To recover the correct leading λ behaviour in (4.19) requires the inclusion of the subleading ν^{-2n} terms displayed in equation (4.21).

Much faster convergence is achieved if we consider the results with the $l = 0$ term removed, as appropriate for electromagnetic modes. Let us illustrate this for the order λ^2 TE mode (now, for simplicity, write $\lambda = \lambda^{\mathrm{TE}}$). Then, in place of the energy (4.19), we have

$$\tilde{E}^{\lambda^2} = \frac{\lambda^2 a}{32\pi} + \frac{\lambda^2 a}{4\pi}\int_0^\infty \frac{\mathrm{d}x}{x^2}\sinh^2 x\, \mathrm{e}^{-2x} = \lambda^2 a\left(\frac{1}{32\pi} + \frac{\ln 2}{4\pi}\right) = \lambda^2 a(0.065\,1061). \tag{5.20}$$

Now the leading term in the uniform asymptotic expansion is no longer zero:

$$\begin{aligned} E^{(0)} &= -\frac{1}{2\pi a}\sum_{l=1}^{\infty}(2l+1)\int_0^\infty \mathrm{d}x\, x\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{\lambda^2 a^2 t^2}{8\nu^2}\right) \\ &= \frac{\lambda^2 a}{8\pi}\sum_{l=1}^{\infty}\nu^0\left(-\frac{\pi}{2}\right) = \frac{\lambda^2 a}{16} = \lambda^2 a(0.0625) \end{aligned} \tag{5.21}$$

which is 4% lower than the exact answer (5.20). The next term in the uniform asymptotic expansion is

$$\begin{aligned} E^{(2)} &= -\frac{\lambda^2 a}{4\pi}[3\zeta(2) - 4]\int_0^\infty \mathrm{d}z\, t^2\frac{t^2 - 6t^4 + 5t^6}{8} \\ &= \lambda^2 a\left(\frac{3\pi^2}{2048} - \frac{3}{256}\right) = \lambda^2 a(0.002\,7368) \end{aligned} \tag{5.22}$$

which reduces the estimate to

$$E^{(0)} + E^{(2)} = \lambda^2 a(0.065\,2368) \tag{5.23}$$

which is now 0.2% high. Further, one more term gives

$$\begin{aligned} E^{(4)} &= -\frac{\lambda^2 a}{8\pi}[15\zeta(4) - 16]\int_0^\infty \mathrm{d}z\, t^2\frac{t^4}{16}(7 - 148t^2 + 554t^4 - 708t^6 + 295t^8) \\ &= -\lambda^2 a\left(\frac{59\pi^4}{524\,288} - \frac{177}{16\,328}\right) = -\lambda^2 a(0.000\,158\,570) \end{aligned} \tag{5.24}$$

and the estimate for the energy is now only 0.04% low:

$$E^{(0)} + E^{(2)} + E^{(4)} = \lambda^2 a(0.065\,078\,23). \tag{5.25}$$

We could also make similar remarks about the TM contributions. However, evidently there are additional subtleties here, so we will defer further discussion to a later publication.

6. Conclusions

In this paper, we have repeated some calculations using 'sharp' but not necessarily 'strong' potentials. That is, we have computed Casimir energies in the presence of $\lambda\delta(x-a)$ potentials, in the cases when the delta function lies on two parallel planes (first considered in [22]), and when the support of the δ function is a sphere (first considered in [23, 24]). We have also considered spherical potentials of the form $\lambda\delta'(r-a)/r$. For either spherical potential, the approach given here yields finite result in all orders, except the third, in powers of the effective coupling constant, $\lambda^{\rm TE}$ or $1/\lambda^{\rm TM}$, respectively. That is, the expression for the energy possesses a logarithmic divergence entirely associated with the order λ^3 Feynman graph. This was rediscovered by Graham *et al* [19, 20], but obscured by the apparent (spurious) divergence they also claimed to find in order λ^2. The bottom line, however, is that these sharp potentials yield a divergent Casimir self-stress.

The generalizations drawn in Graham *et al* papers [17–20] are, however, perhaps too strong. The fact that the $\lambda \to \infty$ limit of the expression for the energy coincides with that for the Dirichlet shell does not prove that the latter is divergent. It does, however, suggest that such an idealization does not yield the full result for the energy of a configuration defined by a real material boundary. This, of course, is no surprise. It has been recognized since at least 1979 [9, 13] that constructing a shell from real materials will yield apparent divergences as the ideal limit is approached; so, for example, a shell of finite thickness made of dielectric material will correspond to a divergent Casimir energy.

So the finite Boyer energy [3] for an ideal sphere results from omitting divergent terms, which may or may not have observable consequences. (It may be, of course, that for electromagnetic modes, the divergence found here could cancel, for which we have provided some evidence.) However, what is remarkable, and of some significance, is that this finite term is unique. For example, Barton has recently exhibited a Buckyball model of a conducting spherical shell that possesses various large energy contributions referring to the material properties of the shell, but which nevertheless possesses a unique, if subdominant, Boyer term of order $1/a$ [25].

It may be useful to compare this situation with a slightly better understood example, the Casimir energy of a dielectric sphere. That is certainly divergent; yet if the divergences are isolated in terms that contribute to the volume and surface energies, in order $(\epsilon-1)^2$ a unique $1/a$ coefficient emerges [10, 11, 23, 34], which may be interpreted as the van der Waals energy [35]. That coefficient diverges in order $(\epsilon-1)^3$ [23]. This fact seems to bear a striking resemblance to the finite Casimir energy found here in order λ^2, and the divergence in the next order. There is also the more than analogous relationship between the finiteness of the Casimir energy for a dielectric–diamagnetic ball with $\epsilon\mu = 1$, and the finiteness found here when $\lambda^{\rm TE}\lambda^{\rm TM} = 4$: in both cases the divergences separately associated with TE and TM modes cancel.

There are also extremely interesting issues related to surface divergences in the local Casimir energy density, which have been discussed recently by Fulling [30]. His ideas will likely have bearing on understanding the nature of the divergences encountered in these problems.

Evidently, there is much work to be done in understanding the nature of quantum vacuum energy. It would obviously be of great benefit if it would be possible to access these questions experimentally.

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Appendix A. Surface energy contribution to the Casimir effect

We first consider the volume integral of the energy density,

$$E_v = \int (\mathrm{d}\mathbf{r})\langle T^{00}\rangle = \frac{1}{2\mathrm{i}}\int (\mathrm{d}\mathbf{r})[\partial^0\partial'^0 + \nabla\cdot\nabla']G(x,x')\Big|_{x'=x}$$
$$= \frac{1}{2\mathrm{i}}\int \mathrm{d}\mathbf{S}\cdot\nabla G(x,x')\Big|_{x'=x} + \frac{1}{2\mathrm{i}}\int (\mathrm{d}\mathbf{r})[\partial^0\partial'^0 - \nabla^2]G(x,x')\Big|_{x'=x}. \tag{A.1}$$

So, apart from irrelevant δ-function contributions (contact terms) coming from the source term of the time Fourier transformed Green function equation

$$[-\nabla^2 - \omega^2 + V(x)]\mathcal{G}(\mathbf{r},\mathbf{r}';\omega) = \delta(\mathbf{r}-\mathbf{r}') \tag{A.2}$$

the surface energy given in (2.19*a*) combines with the volume integral of the local energy density to give

$$E_v + E_s = \frac{1}{2\mathrm{i}}\int (\mathrm{d}\mathbf{r})\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega(t-t')}2\omega^2\mathcal{G}(\mathbf{r},\mathbf{r}';\omega)\Big|_{x'=x} \tag{A.3}$$

which is the precise meaning of (2.16).

The reason the surface energy must be added to the volume energy follows from the local statement of energy–momentum conservation,

$$\partial_\mu T^{\mu\nu} = 0. \tag{A.4}$$

Integrating the time component of this over the volume, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_v + \int \mathrm{d}S_i\langle T^{i0}\rangle = 0. \tag{A.5}$$

The first term here can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (\mathrm{d}\mathbf{r})\int_c \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\tau}(\omega^2 + \nabla\cdot\nabla')\frac{1}{2\mathrm{i}}\mathcal{G}(\mathbf{r},\mathbf{r};\omega)\Big|_{\tau\to 0} \tag{A.6}$$

where τ is the time-splitting between the two field points in the Green function and c is a contour which encircles the singularities on the positive real axis in a positive sense, and those on the negative real axis in a negative sense. Although this time derivative is zero for a static potential, we recognize from its structure that when the surface term in (A.5) is written as

$$\int \mathrm{d}\mathbf{S}\cdot\nabla\frac{1}{\mathrm{i}}\int_c \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{e}^{-\mathrm{i}\omega\tau}(\pm\mathrm{i}\omega)\mathcal{G}(\mathbf{r},\mathbf{r}',\omega)\Big|_{\tau\to 0,\mathbf{r}'=\mathbf{r}} \tag{A.7}$$

where the sign of iω depends on which field the time derivative in $T^{0i} = \partial^0\phi(x)\partial^i\phi(x)$ acts upon, the surface term may also be recognized as a time derivative,

$$\int \mathrm{d}S_i\langle T^{i0}\rangle = \frac{1}{\mathrm{i}}\int \mathrm{d}\mathbf{S}\cdot\nabla\frac{1}{2}(\partial^0+\partial'^0)G(x,x')\bigg|_{x'=x} = \frac{\mathrm{d}}{\mathrm{d}t}E_s \tag{A.8}$$

with

$$E_s = -\frac{1}{2\mathrm{i}}\int \mathrm{d}\mathbf{S}\cdot\nabla G(x,x')\bigg|_{x'=x}. \tag{A.9}$$

The point, in general, is that it is not the volume energy by itself which is a constant of motion, but the sum of the volume and the surface energy. This was first discussed in detail by Romeo and Saharian in the context of Casimir problems with Robin boundary conditions [36, 37], and then in more generality by Fulling [30]. The significance of these terms have been overlooked by many workers in the past largely because they evidently do not contribute to either ideal Dirichlet or Neumann boundary conditions, which are the simple models usually adopted.

Appendix B. Surface energy as bulk energy of boundary layer

Here we show that the surface energy can be interpreted as the bulk energy of the boundary layer. We do this by considering a scalar field in 1 + 1 dimensions interacting with the background

$$\mathcal{L}_{\mathrm{int}} = -\frac{\lambda}{2}\phi^2\sigma \tag{B.1}$$

where

$$\sigma(x) = \begin{cases} h & -\frac{\delta}{2} < x < \frac{\delta}{2} \\ 0 & \text{otherwise} \end{cases} \tag{B.2}$$

with the property that $h\delta = 1$. The reduced Green function satisfies

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \kappa^2 + \lambda\sigma(x)\right]g(x,x') = \delta(x-x'). \tag{B.3}$$

This may be easily solved in the region of the slab, $-\frac{\delta}{2} < x < \frac{\delta}{2}$,

$$g(x,x') = \frac{1}{2\kappa'}\left\{\mathrm{e}^{-\kappa'|x-x'|} + \frac{1}{\hat{\Delta}}[(\kappa'^2-\kappa^2)\cosh\kappa'(x+x') + (\kappa'-\kappa)^2\,\mathrm{e}^{-\kappa'\delta}\cosh\kappa'(x-x')]\right\}. \tag{B.4}$$

Here $\kappa' = \sqrt{\kappa^2+\lambda h}$ and

$$\hat{\Delta} = 2\kappa\kappa'\cosh\kappa'\delta + (\kappa^2+\kappa'^2)\sinh\kappa'\delta. \tag{B.5}$$

This result may also easily be derived from the multiple reflection formulae given in [27]. The energy of the slab now is obtained by integrating the energy density

$$t^{00} = \frac{1}{2\mathrm{i}}(\omega^2 + \partial_x\partial_{x'} + \lambda h)g\bigg|_{x=x'} \tag{B.6}$$

over frequency and the width of the slab. This gives the vacuum energy of the slab

$$E_s = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\kappa}{2\pi}\frac{1}{2\kappa'\hat{\Delta}}\left[(\kappa'-\kappa)^2(-\kappa^2-\kappa'^2+\lambda h)\,\mathrm{e}^{-\kappa'\delta}\delta \right.$$
$$\left. + (\kappa'^2-\kappa^2)(-\kappa^2+\kappa'^2+\lambda h)\frac{\sinh\kappa'\delta}{\delta}\right]. \tag{B.7}$$

If we now take the limit $\delta \to 0$ and $h \to \infty$ so that $h\delta = 1$, we immediately obtain

$$E_s = \frac{1}{2\pi}\int_0^\infty \mathrm{d}\kappa\,\frac{\lambda}{\lambda + 2\kappa} \tag{B.8}$$

which precisely coincides with one-half the constant term in (2.17).

There is no surface term in the total Casimir energy as long as the slab is of finite width, because we may easily check that $\frac{\mathrm{d}}{\mathrm{d}x} g\big|_{x=x'}$ is continuous at the boundaries $\pm\frac{\delta}{2}$. However, if we only consider the energy internal to the slab we encounter not only the energy (2.16) but a surface term from the integration by parts. It is only this boundary term that gives rise to E_s, (B.8), in this way of proceeding.

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